JOURNAL OF MATHEMATICAL ANALYSIS AND APPLICATIONS **114**, 528–547 (1986)

Asymptotic Behavior of Solutions to Polynomial Renewal Equations

KENNETH B. STOLARSKY*

Department of Mathematics, 1409 West Green Street, University of Illinois, Urbana, Illinois 61801

Submitted by R. P. Boas

Many special functions arise as "renormalized" limits of sequences of polynomials that satisfy a polynomial renewal equation. We determine the asymptotic behavior of these sequences of polynomials for both ordinary and "coefficientwise" convergence, and illustrate it with specific examples. © 1986 Academic Press, Inc.

1. INTRODUCTION

An entire function

$$f(x) = \sum_{n=0}^{\infty} c_n x^n \tag{1.1}$$

is often most simply defined as the limit of the Nth partial sums

$$s_N(x) = \sum_{n=0}^{N} c_n x^n \tag{1.2}$$

of its power series expansion about 0. However, this definition need not be the most convenient for establishing various interesting properties of $f(x)$, such as its zero distribution. In his study of the zero distribution of the Bessel function $J_\nu(x)$, Hurwitz [16] made use not of $s_N(x)$, but rather of a sequence of polynomials in x and x^{-1}, the *Lommel polynomials* $R_{m,\nu}(x)$, for which

$$\lim_{m\to\infty} ((\tfrac{1}{2}x)^{\nu+m} R_{m,\nu+1}(x)/\Gamma(\nu+m+1)) = J_\nu(x). \tag{1.3}$$

Set

$$R_{m,\nu+1}(x) = (\tfrac{1}{2}x)^{-m} g_{m,\nu}(\tfrac{1}{4}x^2). \tag{1.4}$$

* Partially supported by NSF Grant MCS-8301615.

0022-247X/86 $3.00

Hurwitz used a Sturmian argument to show that the zeros of $g_{2m,\nu}(x)$ are all positive when $\nu > -1$, and hence that $J_\nu(x)$ has only real roots for $\nu > -1$ (see [25, pp. 302–307]). Here $g_{m,\nu}$ satisfies

$$g_{m+1,\nu}(x) = (\nu + m + 1)\, g_{m,\nu}(x) - x g_{m-1,\nu}(x). \tag{1.5}$$

We shall think of this linear recurrence as a very special case of the (non-normalized) renewal equation.

The objects of this study are the polynomials determined by the renewal equation (3.6). Various examples of special interest are given in Sections 5, 6, 7, and 8. The nature of our results is well illustrated by the following special case.

THEOREM 1. *If $a_0(x) = a_1(x) = 1$ for all x and*

$$a_{n+1}(x) = (\lambda + n)\, a_n(x) + \sum_{k=0}^{n-1} a_k(x)\, x^{n-k} \tag{1.6}$$

for $n \geqslant 1$, then

$$\begin{aligned} \lim_{N\to\infty} \frac{a_N(x)}{a_N(0)} &= \lim_{N\to\infty} \frac{a_N(0)}{(\lambda+1)\cdots(\lambda+N-1)} \\ &= x + 1 + x^{-\lambda}(x + 1 - \lambda)\, e^x \gamma(\lambda + 1, x) \end{aligned} \tag{1.7}$$

where

$$\gamma(\lambda + 1, x) = \int_0^x e^{-s} s^\lambda\, ds \tag{1.8}$$

is the incomplete gamma function.

We shall call the $a_N(x)/a_N(0)$ the "renormalized polynomials." Their convergence is studied in detail in Sections 3 and 4. Note that a slight change in the form of the recurrence may have a strong effect on the nature of the limit. For $a_0(x) = a_1(x) = 1$ and $\lambda = 1$, Theorem 1 yields e^x for the limit, while for

$$a_{n+1}(x) = (1 + n)\, a_n(x) - \sum_{k=1}^{n-1} a_k(x)(-x)^{n-k} \tag{1.9}$$

with the same initial conditions the limit is $1 + 0.5x$. In this case $a_n(0) = n!$, while a simple induction shows $a_n(-2) = 2^{n-1}$; the two rates of growth are quite different, and "renormalization" produces a zero.

Recall that the partial sums $s_N(z)$ are, plainly, determined by the calculation of the derivatives of $f(x)$ at $x = 0$. Theorem 2 of Section 3 shows that the determination of the $a_n(x)$ for a given analytic function $f(x)$ corresponds to the inversion of a certain Laplace transform. This is somewhat less simple, but there should be compensating benefits besides those already obtained in the Bessel case by Hurwitz. For example, numerical experimentation indicates that the zeros of the polynomials $a_n(x)$ lie closer to the zeros of $f(x)$ then do those of $s_n(x)$ for many (though not all) commonly studied special functions.

This study was motivated by (1) the need to study the zeros of certain polynomials $c_m(v)$ generated by a specific quadratic (and hence nonlinear) recurrence (9.2) that arose in connection with a Bessel analogue of the Riemann zeta function [15], and by (2) the very penetrating work [4] in which linear and quadratic recurrence are studied together. We present here a reasonably complete discussion of the first-order asymptotics in the linear case. The nonlinear case presents a vast open area for further study, in which only isolated results are now available. For the Hawkins polynomials of Section 9 we have used facts peculiar to (9.1) and (9.2), together with some results on scalar renewal equations to prove the convergence to $\cos \pi v$. The details are in [23]. The Bernoulli polynomials provide another example of a family of polynomials satisfying a quadratic recurrence. Their zero distribution and asymptotic behavior have rencently been determined with much greater precision than hitherto by Dilcher [5–8]. For the literature on scalar renewal equations, see [3; 9; 10, 285–293, 306–307; 18] and the references therein. Somewhat more remote from our present inquiries are the recurrences treated in [12].

In Sections 3 and 4 we have generalized the $(\lambda+n)$ of (1.6) to $\lambda+\mu n$ where $\mu, \lambda>0$. The situation for $\mu=0$ is significantly different, but in at least one instance it leads to an interesting family of polynomials (see Sect. 8). In (3.10) of Section 3 the reader may take $\gamma_1=\cdots=\gamma_m=0$, but it does seem noteworthy that these extra parameters may be introduced without causing much additional complication in the underlying formalism.

2. Definitions and Notation

We wish to consider power series with arbitrary coefficients, so we shall initially formulate our results in terms of a rather "relaxed" form of convergence. In Section 4 we consider convergence in the ordinary sense.

Let $\mathscr{F}$ be the collection of all formal power series with real coefficients. For $F(x)$ and $F_j(x)$ belonging to $\mathscr{F}$, where $1 \leqslant j < \infty$, we say

$$F_j(x)=\sum_{m=0}^{\infty} c_{mj} x^m \rightarrow \sum_{m=0}^{\infty} c_m x^m = F(x) \tag{2.1}$$

if for any $\varepsilon > 0$ and $M > 0$ there is a $J = J(\varepsilon, M)$ such that

$$|c_{mj} - c_m| < \varepsilon, \qquad 0 \leqslant m \leqslant M, \tag{2.2}$$

provided $j > J$. We say that $F(x)$ is the "coefficientwise limit" of $F_j(x)$, and write

$$(\mathrm{cw}) \lim_{j \to \infty} F_j(x) = F(x). \tag{2.3}$$

For a nice exposition of the theory of formal power series (that incidently studies several quadratic recurrences), see [17]. Henceforth (as in [17]; see also [2]) we shall enlarge $\mathscr{F}$ to include series of the form

$$\sum_{m=m_0}^{\infty} c_m x^m \tag{2.4}$$

where m_0 is any finite (and hence possibly negative) integer.

For any linear operator L and any function $w(t)$ in its range we define a new operator $(L; w(t))$ by

$$(L; w(t)) f(t) = [w(t)]^{-1} L[w(t) f(t)]. \tag{2.5}$$

For any $f(t)$ it is easy to see that

$$\left(\frac{\partial}{\partial t}; t^\alpha \exp\left(\frac{\beta}{t} + \sum_{m=0}^{M} \gamma_m t^m\right)\right) f(t) = \left(\frac{\partial}{\partial t} + \frac{\alpha}{t} - \frac{\beta}{t^2} + \sum_{m=0}^{M} m\gamma_m t^{m-1}\right) f(t), \tag{2.6}$$

so these operators map $\mathscr{F}$ back into $\mathscr{F}$.

3. General Results for Coefficientwise Convergence

We shall determine the coefficientwise limits of inductively defined sequences of polynomials whose definition involves a power series

$$h(x) = \sum_{k=0}^{\infty} h_k x^k, \qquad h_0 = 1. \tag{3.1}$$

We set

$$\frac{xh'(x)}{h(x)} = \sum_{k=1}^{\infty} b_k x^k \tag{3.2}$$

and

$$[h(x)]^\tau = \sum_{k=0}^{\infty} h_k^{(\tau)} x^k. \tag{3.3}$$

For a fixed value of τ we define the exponential generating functions Φ by

$$\Phi(y) = \sum_{k=0}^{\infty} \frac{h_k^{(\tau)}}{k!} y^k. \tag{3.4}$$

Finally, for any real α we set

$$\Psi_\alpha(y) = y^{1-\alpha} \frac{\partial}{\partial y} (y^\alpha \Phi(y)). \tag{3.5}$$

THEOREM 2. *For positive λ and μ, and any scalars a_0, a_1, define $a_n(x) = a_n(x; \lambda, \mu)$ by $a_0(x) = a_0$, $a_1(x) = a_1$, and*

$$a_{n+1}(x) = (\lambda + \mu n)\, a_n(x) + \sum_{k=0}^{n-1} a_k(x)\, b_{n-k} x^{n-k} \tag{3.6}$$

for $n \geqslant 1$. Thus $a_{n+1}(x)$ is a polynomial of degree at most n. Set

$$\tau = 1/\mu \qquad \text{and} \qquad \alpha = a_1 \tau / a_0. \tag{3.7}$$

Then

$$\begin{aligned} a(x) &:= (\mathrm{cw}) \lim_{N \to \infty} \frac{a_N(x)}{a_N(0)} \\ &= \sum_{k=0}^{\infty} \frac{1 + \alpha^{-1} k}{(\lambda + \mu) \cdots (\lambda + k\mu)} h_k^{(\tau)} x^k. \end{aligned} \tag{3.8}$$

COROLLARY. *Under the same hypotheses,*

$$\begin{aligned} &(\mathrm{cw}) \lim_{N \to \infty} \frac{a_N(x)}{\lambda(\lambda + \mu) \cdots (\lambda + (N-1)\,\mu)} \\ &= a_0 \int_0^1 (1-t)^{\tau\lambda - 1}\, \Psi_\alpha(\tau x t)\, dt, \qquad a_0 \neq 0, \\ &= a_1 \tau \int_0^1 (1-t)^{\tau\lambda - 1}\, \Phi(\tau x t)\, dt, \qquad a_0 = 0, \end{aligned} \tag{3.9}$$

if the integrals exist and the series Φ and Ψ_α converge uniformly on compact subsets. If $a_0 = 0$ and $a_1 = 1$ the right-hand side becomes

$$\int_0^\infty e^{-s\lambda} \Phi[\tau x(1 - e^{-s\mu})]\, ds.$$

The corollary is deduced by interchange of integration and summation, writing the beta function integrals in terms of gamma functions, and making the change of variable $t = 1 - e^{-s\mu}$.

If $h(x) = (1-x)^{-1}$ then $b_k = 1$ for all k; hence $\Phi(y) = e^y$ when $\tau = 1$. In this particular case the integral of the corollary is a well-known Laplace transform [20, p. 330] that can be evaluated in terms of the incomplete gamma function, and Theorem 1 quickly follows. The recurrence discussed after Theorem 1 corresponds to $h(x) = 1 + x$.

We begin the proof of Theorem 2 by writing the recurrence in terms of a linear operator.

LEMMA. *The linear recurrence*

$$a_{n+1}(x) = (\lambda + \mu n)\, a_n(x) + \gamma_1 a_{n-1}(x) + \cdots + \gamma_M a_{n-M}(x) + \sum_{k=0}^{n-1} a_k(x)\, b_{n-k} x^{n-k} \tag{3.10}$$

is equivalent to

$$\left(\frac{\partial}{\partial t}; t^{\tau\lambda} h^{\tau}(xt) \exp\left[\frac{\tau}{t} + \sum_{m=1}^{M} \frac{\gamma_m \tau}{m} t^m\right]\right) f(t) + \left(a_0 \frac{\partial}{\partial t} + \frac{a_1 \tau}{t}; h^{\tau}(xt)\right) 1 = 0 \tag{3.11}$$

where $f(t)$ is the formal power series

$$f(t) = \sum_{n=1}^{\infty} a_n(x)\, t^n. \tag{3.12}$$

Proof. Expansion of the operators yields

$$\left(\lambda\tau t^{-1} - \tau t^{-2} + \sum_{m=1}^{M} \tau\gamma_m t^{m-1}\right) f + h^{-\tau}[\tau x h^{\tau-1} h'(xt) f + h^{\tau} f'] + a_0 \tau x \frac{h'(xt)}{h(xt)} + a_1 \tau t^{-1} = 0 \tag{3.13}$$

or

$$f(t) - a_1 t = \lambda t f(t) + \mu t^2 f'(t) + \sum_{m=1}^{M} \gamma_m t^{m+1} f(t) + t[f(t) + a_0] \frac{x t h'(xt)}{h(xt)}. \tag{3.14}$$

The result follows upon equating coefficients of t^{n+1} and using (3.12).

We now let $\gamma_i = 0$ for all i. The coefficient of t^m in $h^\tau(xt)\, f(t)$ is

$$u(m) = \sum_{k=1}^{m} a_k h_{m-k}^{(\tau)} x^{m-k}, \tag{3.15}$$

where the sum is considered to be 0 if $m = 0$. Also,

$$\left(\frac{\partial}{\partial t}; t^{\tau\lambda} \exp(\tau t^{-1})\right) t^s = (\lambda\tau + s)\, t^{s-1} - \tau t^{s-2}. \tag{3.16}$$

Since

$$(L_1; w(t)\, h^\tau) f = (L_2; h^\tau)\, 1 \tag{3.17}$$

is equivalent to

$$(L_1; w(t))(h^\tau f) = L_2(h^\tau), \tag{3.18}$$

we find by equating powers of t^{s-1} in the latter that

$$(\lambda\tau + s)\, u(s) - \tau u(s+1) = -(a_0 s + \tau a_1)\, h_s^{(\tau)} x^s \tag{3.19}$$

or

$$u(s+1) = (\lambda + s\mu)\, u(s) + (a_0 \mu s + a_1)\, h_s^{(\tau)} x^s. \tag{3.20}$$

Iteration yields

$$\begin{aligned} u(s+1) &= (\lambda + s\mu)(\lambda + (s-1)\,\mu)\, u(s-1) \\ &\quad + (\lambda + s\mu)(a_0(s-1)\,\mu + a_1)\, h_{s-1}^{(\tau)} x^{s-1} + (a_0 s\mu + a_1)\, h_s^{(\tau)}\, x^s \end{aligned} \tag{3.21}$$

and a second iteration yields

$$\begin{aligned} u(s+1) &= (\lambda + s\mu)(\lambda + (s-1)\,\mu)(\lambda + (s-2)\,\mu)\, u(s-2) \\ &\quad + (\lambda + s\mu)(\lambda + (s-1)\,\mu)[a_0(s-2)\,\mu + a_1]\, h_{s-2}^{(\tau)} x^{s-2} \\ &\quad + (\lambda + s\mu)[a_0(s-1)\,\mu + a_1]\, h_{s-1}^{(\tau)} x^{s-1} + (a_0 s\mu + a_1)\, h_s^{(\tau)}\, x^s. \end{aligned} \tag{3.22}$$

Since $u(0) = 0$, continued iteration yields

$$\begin{aligned} u(N) &= \sum_{k=1}^{N} a_k h_{N-k}^{(\tau)} x^{N-k} \\ &= \sum_{k=0}^{N-1} (a_0 k\mu + a_1)(\lambda + (k+1)\,\mu) \cdots (\lambda + (N-1)\,\mu)\, h_k^{(\tau)} x^k. \end{aligned} \tag{3.23}$$

Now write

$$a_k = a_k(x) = a_{0k} + a_{1k}x + \cdots + a_{k-1,k}x^{k-1}. \tag{3.24}$$

Upon equating powers of x^0, x^1, x^2,... we find that (suppress the superscript τ for now)

$$\begin{aligned} a_{0N}h_0 &= a_1(\lambda+\mu)\cdots(\lambda+(N-1)\,\mu)\,h_0 \\ a_{0N-1}h_1 + a_{1N}h_0 &= (a_0\mu + a_1)(\lambda+2\mu)\cdots(\lambda+(N-1)\,\mu)\,h_1 \\ a_{0N-2}h_2 + a_{1N-1}h_1 + a_{2N}h_0 &= (2a_0\mu + a_1)(\lambda+3\mu)\cdots(\lambda+(N-1)\,\mu)\,h_2 \end{aligned} \tag{3.25}$$

and in general

$$\sum_{j=0}^{s} a_{j,N-(s-j)}h_{s-j} = (sa_0\mu + a_1)(\lambda+(s+1)\,\mu)\cdots(\lambda+(N-1)\,\mu)\,h_s. \tag{3.26}$$

Recall that $h_0 = h_0^{(\tau)} = 1$. Hence

$$a_{0N} = a_1(\lambda+\mu)\cdots(\lambda+(N-1)\,\mu), \tag{3.27}$$

and

$$\frac{a_{0N-1}}{a_{0N}} = O(N^{-1}). \tag{3.28}$$

Thus

$$\lim_{N\to\infty}\frac{a_{1N}}{a_{0N}} = \frac{a_0\mu + a_1}{a_1(\lambda+\mu)}\,h_1^{(\tau)}. \tag{3.29}$$

It follows that we also have

$$\frac{a_{1N-1}}{a_{0N}} = O(N^{-1}) \tag{3.30}$$

so upon dividing the third line of (3.25) by a_{0N} we find that

$$\lim_{N\to\infty}\frac{a_{2N}}{a_{0N}} = \frac{2a_0\mu + a_1}{a_1(\lambda+\mu)(\lambda+2\mu)}\,h_2^{(\tau)}. \tag{3.31}$$

Upon repeating this procedure we find that

$$\lim_{N\to\infty}\frac{a_{kN}}{a_{0N}} = \frac{ka_0\mu + a_1}{a_1(\lambda+\mu)\cdots(\lambda+k\mu)}\,h_k^{(\tau)}, \tag{3.32}$$

and so

$$a(x)=(\text{cw})\lim_{N\to\infty}\frac{a_N(x)}{a_N(0)}=\sum_{k=0}^{\infty}\frac{ka_0\mu+a_1}{a_1(\lambda+\mu)\cdots(\lambda+k\mu)}h_k^{(\tau)}x^k. \qquad (3.33)$$

This proves the theorem.

4. Ordinary Convergence

Recall

$$xh'(x)=h(x)\sum_{k=1}^{\infty}b_kx^k. \qquad (4.1)$$

It follows from the above that

$$Mh_M=\sum_{k=1}^{M}h_{M-k}b_k \qquad (4.2)$$

and then by an easy induction that

$$b_k=\sum_{i=1}^{p(k)}\theta_ih^{(i)} \qquad (4.3)$$

where the

$$h^{(i)}=h_1^{e_{i1}}\cdots h_k^{e_{ik}}, \qquad e_{ij}\geqslant 0, \qquad (4.4)$$

encode the $p(k)$ unrestricted partitions of k (e.g., $1+1+2+2+2+3=11$ corresponds to $h_1^2h_2^3h_3$). Also,

$$\sum_{i=1}^{p(k)}\theta_i=1, \qquad \sum_{i=1}^{p(k)}|\theta_i|=2^k-1; \qquad (4.5)$$

a useful check is provided by a well-known identity of Euler, that if each h_j is replaced by the number of unrestricted partitions of j, then b_k becomes $\sigma(k)$, the sum of the divisors of k. It follows that

$$|b_k|\leqslant 2^kp(k)\max_{1\leqslant j\leqslant k}|h_j|. \qquad (4.6)$$

Now if

$$|h_j|\leqslant A_1C_1^j \qquad (4.7)$$

for all j, where A_1, C_1 are positive constants independent of j (essentially the statement that $h(x)$ has a nonzero radius of convergence) then

$$|b_k| \leqslant AC_2^k \tag{4.8}$$

for some positive constants A, C_2 independent of k.

THEOREM 3. *If* (4.7) *is valid, then the conclusion of Theorem* 2 *holds in the sense of uniform convergence on compact sets.*

Proof. Fix a compact set S and define

$$\phi_n = \phi_n(x) = \frac{a_n(x)}{((n-1)\,\mu+\lambda)((n-2)\,\mu+\lambda)\cdots(\mu+\lambda)} \tag{4.9}$$

and

$$\Phi_n = \max_{x \in S} |\phi_n(x)|. \tag{4.10}$$

From

$$\phi_{n+1}(x) = \phi_n(x) + \frac{1}{(n\mu+\lambda)\cdots(\mu+\lambda)} \times \sum_{k=1}^{n-1} \phi_k(x)((k-1)\,\mu+\lambda)\cdots(\mu+\lambda)\, b_{n-k} x^{n-k} \tag{4.11}$$

we obtain

$$\Phi_{n+1} \leqslant \Phi_n + \frac{AC^n}{(n\mu+\lambda)\cdots(\mu+\lambda)} \sum_{k=1}^{n-1} \Phi_k \frac{((k-1)\,\mu+\lambda)\cdots(\mu+\lambda)}{C^k} \tag{4.12}$$

where

$$C = C_2 \max_{x \in S} |x|. \tag{4.13}$$

Set

$$B_q = \max_{j \leqslant q} \Phi_j. \tag{4.14}$$

Then

$$B_{q+1} = \max(B_q, \Phi_{q+1}), \tag{4.15}$$

and

$$\Phi_{n+1} \leqslant B_n \left[1 + \frac{AC^n}{(n\mu+\lambda)\cdots(\mu+\lambda)} \sum_{k=1}^{n-1} \frac{((k-1)\mu+\lambda)\cdots(\mu+\lambda)}{C^k} \right]. \tag{4.16}$$

We easily see that for large values of n,

$$\sum_n = \sum_{k=1}^{n} \frac{((k-1)\mu+\lambda)\cdots(\mu+\lambda)}{C^k} \leqslant n\, \frac{((n-1)\mu+\lambda)\cdots(\mu+\lambda)}{C^n} \tag{4.17}$$

since the last term is the largest. Hence for some n_0 depending only on C, and hence only on the set S, we have (by the same reasoning)

$$\begin{aligned}
\sum_{n-1} &\leqslant \sum_{n-2} + \frac{((n-2)\mu+\lambda)\cdots(\mu+\lambda)}{C^{n-1}} \\
&\leqslant \frac{(n-2)((n-3)\mu+\lambda)\cdots(\mu+\lambda)}{C^{n-2}} + \frac{((n-2)\mu+\lambda)\cdots(\mu+\lambda)}{C^{n-1}} \\
&\leqslant \frac{2\max(C, \mu^{-1}C^2)}{C^n}\, ((n-2)\mu+\lambda)\cdots(\mu+\lambda)
\end{aligned} \tag{4.18}$$

for $n \geqslant n_0$. Thus

$$B_{n+1} \leqslant B_n \left[1 + \frac{2A\max(C, \mu^{-1}C^2)}{(n\mu+\lambda)((n-1)\mu+\lambda)} \right], \quad n \geqslant n_0, \tag{4.19}$$

and it follows that the sequence B_n is bounded. Hence

$$|a_n(x)| \leqslant B((n-1)\mu+\lambda)\cdots(\mu+\lambda), \tag{4.20}$$

where the bound B depends only on the set S.

We can now show that the sequence $\phi_n(x)$ converges uniformly on compact sets. It suffices to show that

$$\sum_{n=1}^{\infty} |\phi_{n+1}(x) - \phi_n(x)| \tag{4.21}$$

converges uniformly on compact sets. By the above estimates and (4.11), we see that this sum is dominated by

$$\begin{aligned}
&\sum_{n=1}^{\infty} B \frac{C^n}{(n\mu+\lambda)\cdots(\mu+\lambda)} \sum_{k=1}^{n-1} \frac{((k-1)\mu+\lambda)\cdots(\mu+\lambda)}{C^k} \\
&\quad = \frac{B}{C} \sum_{n=1}^{\infty} \left(\frac{C^n}{(n\mu+\lambda)\cdots(\mu+\lambda)} \sum_{k=1}^{n-1} \frac{((k-1)\mu+\lambda)\cdots(\mu+\lambda)}{C^{k-1}} \right) \\
&\quad = \frac{B}{C} \sum_{s=1}^{\infty} \frac{C^{s+1}}{s\mu(\mu+\lambda)\cdots(s\mu+\lambda)}.
\end{aligned} \tag{4.22}$$

Here the terms have been grouped according to powers of C and summed; see, e.g., [14, p. 316]. Since this converges, the result follows.

5. An Associated Quadratic Recurrence

We now take

$$h(x) = \sum_{k=0}^{\infty} k! x^k = 1 - \exp(-x^{-1}) \int_0^{-x} t^{-1} e^{-1/t}\, dt. \tag{5.1}$$

This formal power series satisfies the differential equation

$$x^2 h'(x) + h(x)(x-1) + 1 = 0 \tag{5.2}$$

and hence (differentiate again) the second-order equation

$$x^2 h''(x) + (3x-1)\, h'(x) + h(x) = 0. \tag{5.3}$$

The change of variable

$$\begin{aligned} y = y(x) &= x \frac{h'(x)}{h(x)} = \sum_{k=1}^{\infty} b_k x^k \\ &= x + 3x^2 + 13x^3 + 71x^4 + 461x^5 + 3447x^6 + \cdots \end{aligned} \tag{5.4}$$

yields the Ricatti equation

$$y' + \left(\frac{2}{x} - \frac{1}{x^2}\right) y + \frac{1}{x} y^2 = -\frac{1}{x}. \tag{5.5}$$

If we set $b_0 = 1$, it follows that the b_i satisfy the quadratic recurrence

$$b_{n+1} = (n+1)\, b_n + \sum_{k=0}^{n-1} b_k b_{n-k}. \tag{5.6}$$

For the polynomials $a_n(x)$ of Section 3 we have (for $\lambda = \mu = 1$, $a_0(x) = a_1(x) = 1$)

$$a_{n+1}(x) = (n+1)\, a_n(x) + \sum_{k=0}^{n-1} a_k(x)\, b_{n-k} x^{n-k}, \tag{5.7}$$

so here

$$a_n(1) = b_n, \qquad n \geqslant 1. \tag{5.8}$$

Calculation yields

$$\begin{aligned}
a_2(x) &= 2 + x \\
a_3(x) &= 6 + 4x + 3x^2 \\
a_4(x) &= 24 + 18x + 16x^2 + 13x^3 \\
a_5(x) &= 120 + 96x + 90x^2 + 84x^3 + 71x^4 \\
a_6(x) &= 720 + 600x + 576x^2 + 558x^3 + 532x^4 + 461x^5.
\end{aligned} \tag{5.9}$$

Here

$$\Phi(y) = (1-y)^{-1} \quad \text{and} \quad \Psi_1(y) = (1-y)^{-2}. \tag{5.10}$$

Hence by the theorem,

$$\text{(cw)} \lim_{N \to \infty} \frac{a_N(x)}{a_N(0)} = \int_0^1 (1-xt)^{-2}\,dt = \frac{1}{1-x}, \tag{5.11}$$

a fact that is indeed suggested by the numerical values of the $a_k(x)$, and that is related to the fact that

$$1 - 1!x + 2!x^2 - 3!x^3 + \cdots = \int \frac{e^{-t}}{(1+xt)}\,dt \tag{5.12}$$

in the Borel sense (see [13, p. 192; 1]. Note that neither the hypothesis nor the conclusion of Theorem 3 is valid here.

6. Exponentials

We already saw after Theorem 1 that for $a_0(x) = a_1(x) = 1$, $\mu = \lambda = 1$, and $h(x) = (1-x)^{-1}$ we have convergence of $a_N(x)/a_N(0)$ to e^x. If we change the value of λ to 0, Theorem 3 with $\alpha = 1$ yields

$$\lim_{N \to \infty} \frac{a_N(x)}{a_N(0)} = (1+x)\,e^x. \tag{6.1}$$

In fact, one can show that

$$a_n(x) = (n-2)!\left[(n-1)\,s_{n-1}(x) + (n-2)\,xs_{n-2}(x) - x^2 s_{n-3}(x)\right] \tag{6.2}$$

for $n \geqslant 2$, where

$$s_n(x) = \sum_{k=1}^{n} \frac{x^k}{k!}. \tag{6.3}$$

It follows that $a_n(-1)=0$ for $n \geqslant 2$. We remark that if

$$a_1(x) = -x - x^2, \qquad a_2(x) = 1 - x^2 \tag{6.4}$$

were the initial conditions, then the polynomial

$$\begin{aligned} a_n(x) &= (n-1)![(n-1)+(n-2)\,x - x^2] \\ &= (n-1)!(x+1)(n-1-x) \end{aligned} \tag{6.5}$$

would solve the recurrence. Note that its nonconstant zero tends to infinity. Here $a_N(x)/a_N(0) \to x+1$.

If $a_0(x)=0$, $a_1(x)=1$, and

$$b_{2m+1}=0, \qquad b_{2m}=(-1)^m\, 2, \tag{6.6}$$

then the polynomials

$$a_{n+1}(x) = na_n(x) - \sum_{k=1}^{n-1} a_k(x)\, b_{n-k} x^{n-k}, \tag{6.7}$$

upon renormalization, *will converge uniformly on compact sets to* $\cos x$. This follows by Theorem 3. These polynomials may be regarded as a type of "Lommel polynomials" for the cosine. Unlike the Lommel polynomials, these can have complex zeros. A short list of them follows:

$$\begin{aligned} a_3(x) &= 2 \\ a_4(x) &= 6 - 2x^2 \\ a_5(x) &= 24 - 10x^2 \\ a_6(x) &= 120 - 54x^2 + 2x^4 \\ a_7(x) &= 720 - 336x^2 + 18x^4 \\ &\ \ \cdots \\ a_{11}(x) &= 3628800 - 1774080x^2 + 131040x^4 - 3360x^6 + 34x^8. \end{aligned} \tag{6.8}$$

Here $a_{2n+1}(x)$ has degree $2n-2$; let $s_{2n-2}(x)$ denote the partial sum of the Maclaurin series for $\cos x$ of degree $2n-2$. Numerical calculations indicate that the roots of $a_{2n+1}(x)$ are closer to those of $\cos x$ than those of $s_{2n-2}(x)$. We list here the smallest (in modulus) roots under the corresponding functions:

n	$\cos x$	s_{2n-2}	a_{2n+1}
3	1.5707963	1.5924504	1.5714897
4	1.5707963	1.5699058	1.5707823
5	1.5707963	1.5708211	1.5807965

Dilcher has pointed out (private communication) that

$$a_{2k+2}(x) = (2k+1)!\, s_{2k}(x) + (2k-1)!\, x^2 s_{2k-2}(x)$$
$$a_{2k+3}(x) = (2k+2)!\, s_{2k}(x) + (2k)!\, x^2 s_{2k-2}(x).$$

7. Bessel Functions

Say $a_0(x) = a_1(x) = 1$, $\mu = 1$, while $b_1 = -1$ and $b_k = 0$ for $k \neq 1$. Then the recurrence is

$$a_{n+1}(x) = (\lambda + n)\, a_n(x) - x a_{n-1}(x), \qquad n \geqslant 1. \tag{7.1}$$

Since $h'/h = -1$ we have

$$h(x) = K_0 e^{-x} \qquad \text{and} \qquad h_k = K_0 \frac{(-1)^k}{k!}, \tag{7.2}$$

so $K_0 = 1$. Thus

$$\Phi(y) = \sum_{k=0}^{\infty} \frac{(-1)^k}{(k!)^2} y^k \tag{7.3}$$

and

$$\Psi_\alpha(y) = \Psi_1(y) = \frac{\partial}{\partial y}(y\Phi(y)) = \sum_{k=0}^{\infty} \frac{(k+1)(-1)^k}{(k!)^2} y^k, \tag{7.4}$$

and Theorem 3 yields

$$\lim_{N\to\infty} \frac{a_N(x)}{a_N(0)} = \sum_{k=0}^{\infty} \frac{(-1)^k (k+1)\, x^k}{k!(\lambda+1)\cdots(\lambda+k)}. \tag{7.5}$$

For $\lambda = 1$ this limit is $J_0(2\sqrt{x})$, and the polynomials are the Lommel polynomials $g_{m,0}(x)$ of order 0. For $\lambda = 0$ the limit is $D_x(xJ_0(2\sqrt{x}))$, and numerical calculations show that the zeros of the polynomials are close to the zeros of this derivative.

8. The Parameter μ

It seems reasonable that a decrease in μ could result in a more rapidly growing limit function. Consider the case $a_0(x)=a_1(x)=1$ and $b_k=1$ for all k. Here it follows that $h_k=1$ for all k, so for $n\geqslant 1$ we have

$$a_{n+1}(x)=(\lambda+\mu n)\,a_n(x)+\sum_{k=1}^{n-1} a_k(x)\,x^{n-k}. \tag{8.1}$$

Set $\tau=1/\mu$; by a standard Laplace transform (see, e.g., [20, p. 330]) we have

$$\lim_{N\to\infty}\frac{a_N(x)}{\lambda(\lambda+\mu)\cdots(\lambda+(N-1)\,\mu)}$$
$$=\int_0^\infty e^{-s\lambda}\Phi[\tau x(1-e^{-su})]\,ds=\frac{\tau e^{\tau x}}{(\tau x)^{\tau\lambda}}\int_0^{\tau x} e^{-u}u^{\tau\lambda-1}\,du. \tag{8.2}$$

We are assuming here that μ, $\lambda>0$. As $x\to\infty$ the integral clearly converges to a finite value, so the limit function in this case is of exponential type at least (in fact, exactly) τ.

We now return to the recurrence (8.1) with $\mu=0$. In this case we have no theorem on convergence, and it becomes conceivable for the $a_n(x)$ to have an arbitrarily large number of zeros in a fixed compact set. We show that with a slight change in initial conditions this will actually happen, and in a quite orderly way.

Theorem 4. *If $\mu=0$ and $a_0(x)=a_1(x)=1$, then all zeros of $a_n(x)$ are $O(|\lambda|)$. If $\lambda\geqslant 1$, they all have modulus exactly λ.*

Proof. Set

$$f(t)=f(\lambda;x,t)=\sum_{n=1}^{\infty} a_n(x)\,t^n. \tag{8.3}$$

Multiplication of both sides of (8.1) by t^{n+1} and summation on n yields

$$f(t)-t=\lambda tf(t)+t\sum_{k=0}^{\infty} a_k(x)\,t^k\sum_{n=k+1}^{\infty}(xt)^{n-k}$$
$$=\lambda tf(t)+\frac{xt^2}{1-xt}\,(f(t)+1); \tag{8.4}$$

here the order of summation has been interchanged. Thus

$$f(t) = \frac{t}{1 - t(x+\lambda) + (\lambda - 1)\,xt^2}$$

$$= \frac{t}{(\lambda - 1)\,x(t - \alpha_1)(t - \alpha_2)} \tag{8.5}$$

where the $\alpha_i = \alpha_i(\lambda, x)$ are the roots of the quadratic in the denominator of f. Partial fractions now yield

$$a_n(x) = \frac{1}{(1-\lambda)\,x} \frac{\alpha_2^{-n} - \alpha_1^{-n}}{\alpha_2 - \alpha_1}. \tag{8.6}$$

Hence an $a_n(x)$ can vanish only if $\alpha_1 = \rho\alpha_2$, where $\rho = e^{i\theta}$ is an nth root of unity. An obvious calculation yields

$$x + \lambda + \sqrt{(x-\lambda)^2 + 4x} = \rho(x + \lambda - \sqrt{(x-\lambda)^2 + 4x}) \tag{8.7}$$

and (after some manipulation)

$$x^2 - 2x[(\lambda - 1)\cos\theta + 1] + \lambda^2 = 0. \tag{8.8}$$

By the quadratic formula the zeros are $O(|\lambda|)$, and for $\lambda \geqslant 1$ it is easily seen that each has modulus λ. We remark that

$$a_n(x) = (x+\lambda)\,a_{n-1}(x) - (\lambda - 1)\,xa_{n-2}(x), \qquad n \geqslant 2, \tag{8.9}$$

and

$$x^n a_{n+1}\left(\frac{\lambda}{x}\right) = a_{n+1}(\lambda x), \qquad n \geqslant 1. \tag{8.10}$$

Somewhat more recondite are the facts that

$$(n-1)\,a_{n+1}(x) = \sum_{k=2}^{n} a_k(x)\,a_{n+2-k}(x) - (\lambda - 1)\,x \times \sum_{k=1}^{n-1} a_k(x)\,a_{n-k}(x), \qquad n \geqslant 2, \tag{8.11}$$

and that if we define polynomials $c_j(\lambda)$ by

$$a_n(x) = \sum_{j=0}^{n-1} c_j(\lambda)\,x^j \tag{8.12}$$

then all zeros of the $c_j(\lambda)$ are real. We reserve the proofs, their combinatorial significance for counting connected components of linear arrays, and various generalizations for a future note.

9. The Hawkins Polynomials

Hawkins [15] has introduced the polynomials defined by

$$a_0(v) = a_1(v) = \tfrac{1}{2}[v^2 - (\tfrac{1}{2})^2] \tag{9.1}$$

and

$$a_{n+2}(v) = \frac{1}{2}\left[(n+3)\, a_{n+1}(v) - \sum_{k=0}^{n} a_k(v)\, a_{n-k}(v)\right], \quad n \geq 0, \tag{9.2}$$

and has found a deep connection between them and the "Bessel" Riemann zeta function. The latter is

$$\zeta_v(s) = \sum_{n=1}^{\infty} \frac{1}{\lambda_{n,v}^s} \tag{9.3}$$

where $\lambda_{n,v}$ is the nth positive zero of the Bessel function $J_v(x)$. For $v = \pm\frac{1}{2}$ we have $\lambda_{n,\pm 1/2} = n\pi$, and for v a half odd integer the corresponding Bessel function is elementary. Heuristic studies by Scroggs [22] indicate that these polynomials have a number of interesting properties, and in particular that *upon renormalization they converge to*

$$\cos \pi v$$

uniformly on compacts sets. This is proved in [23], where it is also established that

$$c_k(v) = \frac{-(k+1)!}{\pi 2^{k+1}}\left[1 + O_S\left(\frac{1}{k}\right)\right] \cos \pi v \tag{9.4}$$

on any compact set S that does not contain half odd integers.

10. Remarks

There is an extensive literature on partial sums of power series and their zeros. See, for example, [11, 21, 24] and the references therein. However, the classical Lommel polynomials already demonstrate that many of the results in [11] on the angular distribution of zeros for $s_N(x)$ cannot be directly extended to the $a_N(x)$.

The use of generating functions is a prominent feature of our present study of polynomial sequences generated by linear recurrences. It is worth mentioning that the paper [19] of Pollaczek makes a very profound use of this technique in the case of three term recurrences.

REFERENCES

1. E. J. BARBEAU, Euler subdues a very obstreperous series, *Amer. Math. Monthly* **86** (1979), 356–372.
2. E. A. BENDER, A lifting theorem for formal power series, *Proc. Amer. Math. Soc.* **42** (1974), 16–22.
3. J. S. CHIPMAN, A renewal model of economic growth: The discrete case, *in* "Mathematical Topics in Economic Theory and Computation" (R. H. Day and S. M. Robinson, Eds.), Soc. Indus. Appl. Math., Philadelphia, 1972.
4. N. G. DEBRUIJN AND P. ERDÖS, Some linear and some quadratic recursion formulas, *Nederl. Akad. Wetensch. Proc. Ser. A*, I **54** (1951), 374–382; II **55** (1952), 152–163.
5. K. DILCHER, "Zeros of Bernoulli polynomials," Doctoral thesis, Queen's University, Ontario, 1983.
6. K. DILCHER, The maximum modulus of zeros of polynomials, *C. R. Math. Rep. Acad. Sci. Canada* **5** (1983), 189–194.
7. K. DILCHER, Zero-free regions for Bernoulli polynomials, *C. R. Math. Rep. Acad. Sci. Canada* **4** (1983), 241–246.
8. K. DILCHER, Zero-free regions for Bernoulli polynomials, in preparation.
9. P. ERDÖS, W. FELLER, AND H. POLLARD, A property of power series with positive coefficients, *Bull. Amer. Math. Soc.* **55** (1949), 201–204.
10. W. FELLER, "An Introduction to Probability Theory and its Applications," 2nd ed., Wiley, New York, 1960.
11. T. GANELIUS, Sequences of analytic functions and their zeros, *Ark. Mat.* **3** (1954), 1–50.
12. I. GUMOWSKI AND C. MIRA, "Recurrences and Discrete Dynamic Systems," Lecture Notes in Mathematics Vol. 809, Springer–Verlag, Berlin, 1980.
13. G. H. HARDY, "Divergent Series," Oxford Univ. Press (Clarendon), London/New York, 1963.
14. H. S. HALL AND S. R. KNIGHT, "Higher Algebra," 4th ed., MacMillan & Co., London, 1891; reprinted 1964.
15. J. HAWKINS, "On a Zeta Function Associated with Bessel's Equation," Doctoral thesis, University of Illinois, 1983.
16. A. HURWITZ, Ueber die Nullstellen der Bessel'schen Function, *Math. Ann.* **33** (1889), 246–266.
17. I. NIVEN, Formal power series, *Amer. Math. Monthly* **76** (1969), 871–889.
18. M. A. PINSKY, A note on the Erdös–Feller–Pollard theorem, *Amer. Math. Monthly* **83** (1976), 729–731.
19. F. POLLACZEK, Sur une généralisation des polynomes de Jacobi, *Mem. Sci. Math.* **131** (1956).
20. G. E. ROBERTS AND H. KAUFMAN, "Table of Laplace Transforms," Saunders, Philadelphia, 1966.
21. E. B. SAFF AND R. S. VARGA, On the zeros and poles of Padé approximants to e^z, III, *Numer. Math.* **30** (1978), 241–266.

22. J. SCROGGS, private communication.
23. K. B. STOLARSKY, Singularities of Bessel-zeta functions, and Hawkins' polynomials, *Mathematika* **32** (1985), 96–103.
24. R. S. VARGA, "Topics in Polynomial Interpolation and Approximation," Montreal Univ. Press, Montreal, 1982.
25. G. N. WATSON, "A Treatise on the Theory of Bessel Functions," 2nd ed., Cambridge Univ. Press, Cambridge, 1966.